Earth's density flattening and hypothesis of latitudinal normal density

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Abstract In this paper, the definition of latitudinal density and density flattening of the level ellipsoid is given, and integral formulas of latitudinal density for pole gravity and equator gravity are derived. According to the pole gravity condition and equator gravity condition for the level ellipsoid, latitudinal density distribution function of the level ellipsoid is obtained. It is proved mathematically that latitudinal density of the earth's equator is larger than that of the pole, the earth's density flattening calculated preliminarily is 1/322, and hypothesis of the earth's latitudinal normal density is further proposed, so that theoretical preparation for studying the forming cause of the earth gravity in problems such as continent drift, mantle convection, and submarine extension is made well.

Keywords: latitudinal density, density flattening, hypothesis of Earth's latitudinal normal density.

In 1687, Newton firstly raised the viewpoint, the earth being an ellipsoid. It was proved mathematically that the earth was an oblate ellipsoid with equator's radius larger than pole's, the earth's geometrical flattening $f=(a-b)/a$ estimated was 1/239. But in 1716, Cassini and his son, in the light of Picard's and their own not precise arc measurement results, concluded that the earth was a prolate ellipsoid with pole's radius larger than equator's. In this way, in scientific history occurred an argument about oblate theory and prolate theory. To solve this argument, in 1735, Paris Academy of Sciences sent two teams, one survey team went to Lappland near the Arctic Circle, the other got to Kito nearby equator to carry out the precise arc measurements. The results showed that per arc length of meridian line in equator neighborhood was minimum, middle in Paris and maximum in Arctic neighborhood. It is fully testified that oblate theory is correct. Newton's oblate theory is correct, but the flattening value 1/239 he calculated was not exact. Following Newton, Huygens gave another flattening value 1/577, and people attempted to interpret its difference in theory and observation, but failed. Afterward, many people gave the flattening values, for instance, in 1789 Legendre's 1/318, in 1841 Bessel's 1/299, in 1866 Clarke's 1/295, in 1909 Hayford's 1/297, in 1948 Bullard's 1/297.34. Now, the geometrical flattening of the earth has been affirmed.

Researches on the earth's figure and the earth's density are basic tasks in earth gravity, the theory and method concerning the earth's figure have been fully developed, but, in comparison with that of the earth's density, it seems to be rather weak. Traditional Stokes' theory and Molodensky's theory are based on the prerequisite of mathematics evading density distribution to carry out the research on the earth's figure, the earth's figure theory almost becomes the synonym for earth gravity. With the rapid development of space geodetic techniques and the continuous perfection of the earth's figure theory, the earth figure should not be constant theoretical subject of earth gravity. For there exist the problems of integral density distribution forms of the earth, such as continent drift, mantle convection and submarine extension, the focal point of researches on earth gravity is requested to transfer from earth figure to the earth's density.

Earth gravity theory for studying the earth's density was firstly put forward by Clairaut, and in 1743, he published the equilibrium figure theory. In 1825, Legendre and Laplace, in terms of this theory, obtained the internal density law of the earth. Afterward, in 1884 Darwin, in 1888 Roche, in 1897 Wiechert, in 1945 Bullard and in 1975 Bullen separately got the similar density law. But these density laws obtained are radial density distribution $\delta = \delta(r)$, latitudinal density distribution $\delta = \delta(B)$ of the earth has not been available, thus leaving a gap in studying the latitudinal density for the earth gravity theory.

Studying the latitudinal density distribution of the earth is significant in recognizing the latitudinal displacements of the earth's matter, for approaching the forming causes of earth gravity in question such as continent drift, mantle convection and submarine extension, the relation of latitudinal distribution of earth's matter and latitudinal displacement is more direct than that of radial distribution of the earth's matter and radial displacement.

As corresponding relations of the earth's density and latitude had not been determined yet in predecessors' researches, to find a new start, in terms of research history of the earth's geometrical flattening, we wonder whether there exists density flattening in the earth. If yes, then how to express the earth's density flattening remains unresolved.

Let the inside of the ellipsoid have a resembling ellipsoid group with equal flattening, this group is continuously contracted from surface of the ellipsoid to the center of the ellipsoid. A radial of any point on the ellipsoid surface passes through all resembling ellipsoid surfaces of this group to the center of the ellipsoid, and latitude on every resembling ellipsoid for all points at the radial is the same. So we define that mean value of density of all points on the radial is called the latitudinal density at this point of the ellipsoid.

It can be seen from the definition mentioned above that the concept of latitudinal density makes density inside the ellipsoid abstract (not compress) to the surface of the ellipsoid, and a correspondence relationship $\delta = \delta(B)$ between density and latitude is set up. It is possible for us to utilize the concise mathematical method to quantitatively study the latitudinal distribution of the earth's density.

Since there is a concept of latitudinal density, there is a concept of density flattening. If δ_E and δ_P are latitudinal density of equator and pole respectively, then density flattening of the ellipsoid is $f' = (\delta_E - \delta_P)/\delta_E$. According to the close interrelation tradition of geometrical quantity and physical quantity in earth gravity, there is an argument about oblate theory and prolate theory corresponding to the earth's geometrical flattening. Similarly, there exists a difference between prolate theory ($\delta_E < \delta_P$) and oblate theory ($\delta_E > \delta_P$) about the earth's density flattening.

This paper derives the integral formulas of latitudinal density for pole gravity and equator gravity. In the light of pole gravity condition and equator gravity condition for the level ellipsoid, solution of latitudinal density distribution function for the level ellipsoid is made. It is proved mathematically that latitudinal density of the earth's equator is larger than that of the pole ($\delta_E > \delta_P$), in the meantime, the earth's density flattening preliminarily calculated is 1/322, and hypothesis of the earth's latitudinal normal density is put forward.

1 Latitudinal density distribution function

In order to study latitudinal density of the earth we should first investigate latitudinal density of the level ellipsoid. Moritz detected that some rational matter distribution inside the level ellipsoid is unknown, but it must have an inhomogeneous, non-equilibrium matter distribution[1]. In 1978, Iona's researches indicated that homogeneous ellipsoid could not be the same as the level ellipsoid, and produced so great gravity difference between pole and equator[2]. In 1996, Maialle and Hipolito used the layering-homogeneous ellipsoid to compute gravity in pole and equator, which was compared with the level ellipsoid, and got good results[3]. In 1996, Hao Xiaoguang expressed that inhomogeneous density of the level ellipsoid could be distributed on the basis of latitude[4].

In 1894, Pizzetti divided the level ellipsoid into "inner body" and "surface layer", the former is Maclaurin ellipsoid, the latter is a single layer surface in which Maclaurin ellipsoid is wrapped tightly[1]. Maclaurin ellipsoid is a homogeneous, equipotential ellipsoid, and its mass M_{MC} and angular velocity ω must satisfy the following relation[5]

$$M_{MC} = \frac{2\omega^2 a^3 e'^3}{3G((3+e'^2)\mathrm{arctg}\ e' - 3e')\sqrt{1+e'^2}}, \tag{1}$$

where $e'^2 = \dfrac{a^2-b^2}{b^2}$, G is universal gravitation constant.

Substituting the constants (a, b, ω) of 1980 geodetic reference system into eq. (1) to carry out calculation, M_{MC} obtained will be larger than mass of the level ellipsoid M. Thus, in Pizzetti's model, mass of the surface layer is negative, magnitude of its "negative mass" just offsets the surplus mass (M_{MC}–M) of the inner body. The formula[5] of surface layer density distribution is as follows:

$$\mu(B)=\frac{(M-M_{MC})\sqrt{a^2\cos^2 B+b^2\sin^2 B}}{4\pi a^2 b},\tag{2}$$

in pole and equator,

$$\mu_P=\mu(90^\circ)=\frac{M-M_{MC}}{4\pi a^2},\ \mu_E=\mu(0^\circ)=\frac{M-M_{MC}}{4\pi ab}.\tag{3}$$

Substituting (3) into (2), we have

$$\mu(B)=-\sqrt{\mu_E^2\cos^2 B+\mu_P^2\sin^2 B}.\tag{4}$$

Owing to that the surface layer of Pizzetti's model is negative density, this is unrealistic in physics. So we assume that the latitudinal density model defined in this paper is used to study the density distribution of the earth.

Pizzetti's model is unrealistic in physics, but it is very successful in mathematics. The latitudinal density model in this paper is based on Pizzetti's model, and has been improved. As Maclaurin ellipsoid is homogeneous ellipsoid, its density does not vary with latitude, therefore functional form of latitudinal density does not interrelate to the inner body of Pizzetti's model, only interrelates to its surface layer. Since the surface layer density of Pizzetti's model is based on latitudinal distribution, and latitudinal density is also based on latitudinal distribution, we assume that functional form of latitudinal density carries on the functional form of surface layer of Pizzetti's model. Then comparing with eq. (4) we get

$$\delta(B)=\sqrt{\delta_E^2\cos^2 B+\delta_P^2\sin^2 B}.\tag{5}$$

Eq. (5) is the functional form of latitudinal density, where δ_E and δ_P are two waiting constants, i.e. latitudinal density in equator and latitudinal density in pole. The difference between eqs. (5) and (4) is that $\mu(B)$ denotes surface density but $\delta(B)$ denotes body density. The latitudinal density $\delta=\delta(B)$ is density mean value of all points on a radial of the ellipsoid, also an equivalent density. In addition, the surface of Pizzetti's model is equipotential surface, but eq. (5) cannot guarantee this. For this reason, we regard level ellipsoid pole and equator gravity as restrictive condition to calculate the waiting constants δ_E and δ_P, so that latitudinal density model approximates to level ellipsoid.

2 Integral formula for latitudinal density of pole gravity

In eq. (5), B is geographic latitude, if u is complementary angle of reduced latitude, substituting by trigonometric function, we change eq. (5) as follows (fig. 1):

$$\delta(u)=\sqrt{\frac{\delta_E^2 b^2\sin^2 u+\delta_P^2 a^2\cos^2 u}{b^2\sin^2 u+a^2\cos^2 u}}.\tag{6}$$

Let the original point of spherical coordinate system $P-r\psi\lambda$ be based on ellipsoid pole, minor axis of the ellipsoid be coordinate axis about ψ, then ellipsoid equation is

$$r_0 = \frac{2b(1+e'^2)\cos\psi}{1+e'^2\cos^2\psi}. \quad (7)$$

As centrifugal force at pole is zero, pole gravity is

$$\gamma_p = G\int_\tau \delta \frac{r\cos\psi}{r^3}\mathrm{d}\tau$$

$$= 4\pi Gb(1+e'^2)\int_0^{\frac{\pi}{2}} \delta \frac{\cos^2\psi\sin\psi}{1+e'^2\cos^2\psi}\mathrm{d}\psi. \quad (8)$$

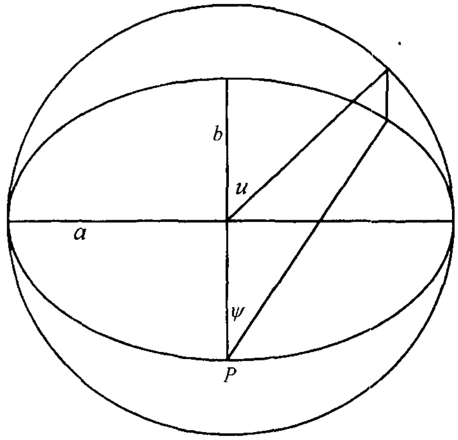

Fig. 1. Polar coordinate system.

Substituting by trigonometric function, we have

$$\gamma_p = 2\pi Gb(1+e'^2)\left(\int_0^1 \frac{\delta(x)\ (1+x)^2\mathrm{d}x}{(e'^2(1-x^2)+2x+2)^{\frac{3}{2}}} + \int_0^1 \frac{\delta(x)\ (1-x)^2\mathrm{d}x}{(e'^2(1-x)^2-2x+2)^{\frac{3}{2}}}\right). \quad (9)$$

In eq. (9), $x=\cos u$, and $\delta(x)=\delta(\cos u)$ is given from eq. (6),

$$\delta(x) = \sqrt{\frac{\delta_E^2 b^2(1-x^2)+\delta_P^2 a^2 x^2}{b^2(1-x^2)+a^2x^2}}. \quad (10)$$

3 Integral formula for latitudinal density of equator gravity

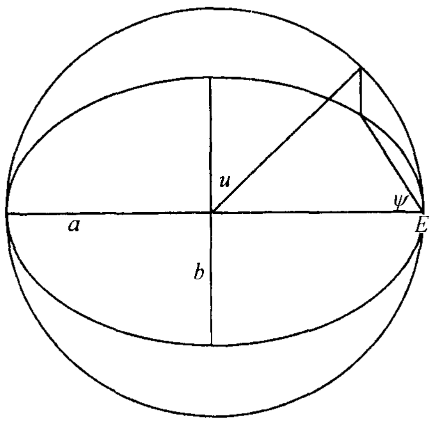

Fig. 2. Equator coordinate system.

Let the original point of spherical coordinate system $E-r\psi\lambda$ be based on ellipsoid equator, major axis of the ellipsoid be coordinate axis about ψ. Then the ellipsoid equation is (fig. 2)

$$r_0 = 2a\cos\psi(e'^2\sin^2\psi\sin^2\lambda-1), \quad (11)$$

and equator gravity is

$$\gamma_e = G\int_\tau \delta\frac{r\cos\psi}{r^3}\mathrm{d}\tau - \omega^2 a$$

$$= 4\pi Ga\int_0^{\frac{\pi}{2}} \delta \frac{\cos^2\psi\sin\psi}{\sqrt{1+e'^2\sin^2\psi}}\mathrm{d}\psi - \omega^2 a. \quad (12)$$

Substituting by trigonometric function, we have

$$\gamma_e = 4\pi G\frac{b^2}{a}\left(\int_0^1 \frac{\delta(y)\ A_W^{\frac{5}{2}}\mathrm{d}y}{(1+y)^2\sqrt{1+e'^2-e'^2A_W}} + \int_0^1 \frac{\delta(y)\ A_E^{\frac{5}{2}}\mathrm{d}y}{(1-y)^2\sqrt{1+e'^2-e'^2A_E}}\right) - \omega^2 a, \quad (13)$$

where $A_W = \dfrac{(1+e'^2)(1+y)}{2+e'^2(1+y)}$, $A_E = \dfrac{(1+e'^2)(1-y)}{2+e'^2(1-y)}$.

In eq. (13), $y = \sin u$, and $\delta(y) = \delta(\sin u)$ is given from eq. (6),

$$\delta(y) = \sqrt{\frac{\delta_E^2 b^2 y^2 + \delta_P^2 a^2 (1-y^2)}{b^2 y^2 + a^2(1-y^2)}}. \tag{14}$$

4 Solution of waiting constants

Let $\kappa = \delta_P / \delta_E$, from eqs. (10) and (14), we get

$$\delta(x) = \delta_E \sqrt{\frac{b^2(1-x^2)+\kappa^2 a^2 x^2}{b^2(1-x^2)+a^2x^2}} = \delta_E \kappa_x, \tag{15}$$

$$\delta(y) = \delta_E \sqrt{\frac{b^2 y^2 + \kappa^2 a^2 (1-y^2)}{b^2 y^2 + a^2(1-y^2)}} = \delta_E \kappa_y. \tag{16}$$

Substituting eqs. (15) and (16) into eqs. (9) and (13), we have

$$\frac{\gamma_P}{1+e'^2}\left(\int_0^1 \frac{\kappa_y A_W^{\frac{5}{2}} \mathrm{d}y}{(1+y)^2\sqrt{1+e'^2-e'^2 A_W}} + \int_0^1 \frac{\kappa_y A_E^{\frac{5}{2}} \mathrm{d}y}{(1-y)^2\sqrt{1+e'^2-e'^2 A_E}}\right) =$$

$$= \frac{a(\gamma_e + \omega^2 a)}{2b}\left(\int_0^1 \frac{\kappa_x (1+x)^2 \mathrm{d}x}{(e'^2(1-x^2)+2x+2)^{\frac{3}{2}}} + \int_0^1 \frac{\kappa_x (1-x)^2 \mathrm{d}x}{(e'^2(1-x)^2-2x+2)^{\frac{3}{2}}}\right), \tag{17}$$

$$\delta_E = \frac{\gamma_P}{2\pi b G(1+e'^2)}\left(\int_0^1 \frac{\kappa_x (1+x)^2 \mathrm{d}x}{(e'^2(1-x^2)+2x+2)^{\frac{3}{2}}} + \int_0^1 \frac{\kappa_x (1-x)^2 \mathrm{d}x}{(e'^2(1-x)^2-2x+2)^{\frac{3}{2}}}\right)^{-1}. \tag{18}$$

Connecting eq. (17) with eq. (18), let

$$a = 637813700\,\mathrm{cm},\ b = 635675200\,\mathrm{cm},\ \omega = 7292115\times 10^{-11}/s$$

satisfy the 1980 geodetic reference system, and

$$\gamma_p = 983.218637\,\mathrm{cm}/s^2,\ \gamma_e = 978.032726\ \mathrm{cm}/s^2$$

are the restrictive condition of connective equations, and gravity value of level ellipsoid pole and equator[4]. Then, by use of numerical integral method, from connective eqs. (17) and (18), the waiting constants can be calculated as follows:

$$\delta_E = 5.526625\ \mathrm{g/cm^3},\ \delta_P = 5.509460\,\mathrm{g/cm^3}. \tag{19}$$

From eq. (19), we can get density flattening of the level ellipsoid in the following

$$f' = (\delta_E - \delta_P)/\delta_E = 1/322. \tag{20}$$

Substituting eq. (19) into eq. (5), we can obtain latitudinal density distribution function of the level ellipsoid as follows:

$$\delta(B)=\sqrt{5.526625^2\cos^2 B+5.509460^2\sin^2 B}. \tag{21}$$

5 Discussion

The restrictive condition of connective equations is only gravity value of level ellipsoid pole and equator, not gravity value of whole level ellipsoid. So density flattening value 1/322 given in eq. (20) is not exact. At present, this work has just started, with the deepening of research work, more and more precise density flattening value will be obtained. As described in the first part of this paper, the geometrical flattening value computed firstly may not be exact. In fact, in process of unceasingly solving geometrical flattening are produced some new theories which greatly promote the development in earth gravity. We wish that in carrying out the solution of density flattening, we could get new theory in order to make contributions to earth gravity.

According to the tradition of earth gravity, latitudinal density of the level ellipsoid can be regarded as latitudinal normal density of the earth, like gravity of the level ellipsoid acts as normal gravity of the earth. So eq. (21) means that latitudinal normal density of the earth is based on latitude distribution, which contracts to equator from pole, the higher the latitude is, the lower the latitudinal density is, this is the inherent properties of the earth. We call it hypothesis of latitudinal normal density of the earth.

By the hypothesis mentioned above, if latitudinal density of the earth is normally distributed on the basis of eq. (21), then mass distribution state of the earth is steady, and if latitudinal density of the earth is anomalously distributed, then mass distribution state of the earth is not steady, and the earth's mass will be displaced and regulated to reach the demand of eq. (21). It can be seen from this point that hypothesis of latitudinal normal density of the earth we proposed is theoretical preparation for studying and approaching the forming cause for the earth gravity in question such as continent drift, mantle convection and submarine extension.

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References

1. Moritz,H., The Figure of the Earth: Theoretical Geodesy and the Earth's Interior (in Chinese), translated by Chen Junyong and Zuo Chuanhui, Beijing: Surveying and Mapping Publishing House, 1992, 92—107.
2. Iona,M., Why is g larger at the poles? Am. J. Phys., 1978, 46: 790—791.
3. Maialle, M. Z., Hipolito, O., Acceleration of gravity for the earth model as an ellipsoidal mass with nonuniform density, Am. J. Phys., 1996, 64: 434—436.
4. Hao Xiaoguang, Modification of conception of latitude correction in gravity measurement, Crustal Deformation and Earthquake (in Chinese), 1996, 16(3): 8—13.
5. Guo Junyi, The Fundamental of Physical Geodesy (in Chinese), Wuhan: Wuhan Technology University of Surveying and Mapping Press, 1994, 135—136.